XI. *Of the Fluents of* Multinomials, *and Series affected by* radical Signs, *which do not begin to converge till after the ſecond Term; in a Letter from* T. Simpſon *F. R. S. to* W. Jones *Eſq; V. P. R. S.*

Preſented May 26. 1748.

ALTHO' the Application of infinite Series, and the Quadrature of the conic Sections, to the inverſe Method of Fluxions has exerciſed the Pens of the moſt able Mathematicians, and produced many curious and uſeful Diſcoveries, yet nothing has been hitherto given, that I know of, whereby the Fluents of radical Multinomials and Series, which do not begin to converge till after the ſecond Term, can be determined, ſo as to be of Uſe in the Solution of Problems: The common Method, by expanding the given Expreſſion, being, you know, altogether impracticable in this Caſe.

The Conſideration of which induced me to draw up the following Paper; which I humbly beg Leave to lay before you, who are ſo good a Judge of the various Improvements which this Subject has from time to time received.

What moſt encourages me to hope this little Eſſay will meet with your Approbation, is, that it is not merely an abſtracted uſeleſs Speculation, but may be apply'd to good purpoſe in many difficult and important Enquiries into Nature; whereof I have put down one or two Inſtances, and ſhall further take the Liberty to obſerve here, that moſt of the lunar

lunar Equations, given by Sir *Iſaac Newton*, are only ſuch Approximations as may be exhibited by the firſt Term of a Series derived by the Method here delivered.

Propoſition.

The Fluent of $\overline{a+cz^{n}}|^{m} \times z^{pn-1}\dot{z}$ *being given (either in algebraic Terms, or from the Quadrature of the Conic Sections,* &c.) *it is propoſed, by means thereof, to approximate the Fluent of* $\overline{a+cx^{n}+dx^{2n}+ex^{3n}+fx^{4n}\ \&c.}|^{m} \times x^{pn-1}\dot{x}$; *ſuppoſing the Series not to converge till after the ſecond Term.*

Make $cz^{n}=cx^{n}+dx^{2n}+ex^{3n}$ &c. and let Q be the given Fluent of $\overline{a+cz^{n}}|^{m} \times z^{pn-1}\dot{z}$, anſwering to any propoſed Value of x: Moreover let $y=x^{pn}$, or $y^{\frac{1}{p}}=x^{n}$, and let this Value of x^{n} be ſubſtituted in the firſt Equation, and it will become $cz^{n}=cy^{\frac{1}{p}}+dy^{\frac{2}{p}}+ey^{\frac{3}{p}}$ &c. whereof the Root y being extracted, we ſhall (by making $R=-\frac{pd}{c}$, $S=\frac{p.\overline{p+3}}{2}\times\frac{d^{2}}{c^{2}}-\frac{pe}{c}$, $T=\frac{-p.\overline{p+4}.\overline{p+5}}{6}\times\frac{d^{3}}{c^{3}}+\frac{p.\overline{p+4}}{1}\times\frac{de}{c^{2}}-\frac{pf}{c}$ &c.) have $y\ (x^{pn})=z^{pn}+Rz^{pn+n}+Sz^{pn+2n}$ &c. whence we alſo obtain $x^{pn-1}\dot{x}=z^{pn-1}\dot{z}+\frac{p+1}{p}\times Rz^{pn+n-1}\dot{z}$ $+\frac{p+2}{p}\times Sz^{pn+2n-1}\dot{z}$ &c.

Let this Value, with that of $cx^{n}+dx^{2n}+ex^{3n}$ &c. (above given) be now ſubſtituted in the propoſed

posed Fluxion, and it will become $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z}$ &c.

Moreover, let v denote the Place, or Distance, of any Term, of this Expression, from the first (exclusive) then the Term itself (drawn into the common Multiplicator) will be denoted by $\overline{a+cz^n}|^m \times \frac{p+v}{p} \times Az^{pn+vn-1}\dot{z}$; and the Fluent thereof will be truly expressed by $\frac{p+1}{p+m+1} \times \frac{p+2}{p+m+2} \times \frac{p+3}{p+m+3} \times \ldots \frac{p+v}{p+m+v} \times \overline{\frac{a}{-c}}|^v \times AQ + \frac{\overline{p+v}.A}{p} \times \frac{\overline{a+cz^n}|^{m+1} \times z^{pn-n}}{\overline{p+m+v} \times nc}$ into $z^{vn} - \frac{p+v-1}{p+m+v-1} \times \frac{az^{vn-n}}{c} + \frac{\overline{p+v-1}.\overline{p+v-2}}{\overline{p+v+m-1}.\overline{p+v+m-2}} \times \frac{a^2 z^{vn-2n}}{c^2}$ &c. continued to as many Terms as there are Units in v. Wherein let v be expounded by 1, 2, 3 &c. successively, and R, S, T, &c. by A respectively: By which means the Fluent of the whole Expression will be obtained.

Corol. 1.

Because the Fluent of the general Term, when the Multiplicator $\overline{a+cz}|^{m+1}$ becomes $=0$, is barely $= \frac{p+1}{p+m+1} \times \frac{p+2}{p+m+2} \times \frac{p+3}{p+m+3} \times \ldots \frac{p+v}{p+m+v} \times \overline{\frac{a}{-c}}|^v \times AQ$ the Fluent of the whole Expression will, therefore, in this Case be truly defined by

$\mathcal{Q} \times 1 - \frac{p+1 . Ra}{p+m+1 . c} + \frac{p+1 . p+2 . S2a}{p+m+1 . p+m+2 . c^2} - \frac{p+1 . p+2 . p+3 . Ta^3}{+m+1 . p+m+2 . p+m+3 . c^3}$, &c. Where $\mathcal{Q}$ denotes the Fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$, when $z^n = \frac{a}{-c}$.

Corol. 2.

But, if $m+1$ and p be, each of them, the Half of an odd affirmative Number, and $\mathcal{P}$ be taken to denote the Periphery of a Circle whose Diameter is Unity, and $-c$ be put $= b$, then the Valne of $\mathcal{Q}$ (or the Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $z^n = \frac{a}{b}$) will be $= \frac{a^{p+m} \mathcal{P}}{nb^p} \times$

$$\frac{1 . 3 . 5 . 7 \ \&c. \ (\text{to } p-\tfrac{1}{2} \text{ Factors}) \times 1 . 3 . 5 . 7 \ \&c. \ \text{to } (m+\tfrac{1}{2} \text{ Factors})}{2 . 4 . 6 . 8 . 10 . 12 \ \&c. \ (\text{to } p+m \text{ Factors})}$$

Therefore the *Whole*, required, Fluent, of $\overline{a-bx^n+dx^{2n}+ex^{3n}\&c.}|^m \times x^{pn-1}\dot{x}$ is, in this Case, equal to the Product of that Expression into the following Series, $1 + \frac{p+1 . Ra}{p+m+1 . b} + \frac{p+1 . p+2 . Sa^2}{p+m+1 . p+m+2 . b^2}$ &c. Wherein R is to be taken $= \frac{pd}{b}$, $S = \frac{p . p+3}{2} \times \frac{d^2}{b^2} + \frac{pe}{b}$, $T = \frac{p . p+4 . p+5}{6} \times \frac{d^3}{b^3} + \frac{p . p+4}{1} \times \frac{de}{b^2} + \frac{pf}{b}$, &c. according to what is above specified.

The Use of what has been deliver'd above will, in some measure, appear from the Solution of the two

two following Problems, which I ſhall ſubjoin as Examples thereof. The firſt is;

To find the Time of Oſcillation in the Arch of a Cycloid, in a Medium reſiſting according to the duplicate Ratio of the Velocity.

Let A denote the whole Arch of the Semi-Cycloid, or the Length of the Pendulum, a the the Arch deſcribed in the whole Deſcent, and x any variable Part thereof deſcribed from the Beginning of the Deſcent; and let the Denſity of the Medium be, every-where, as $\frac{1}{b}$: Then the Fluxion of the Time will be found =

$$\overline{a - 1 + \frac{2a}{b} \times \frac{x}{2} - \frac{2x^2}{2.3b} + \frac{4x^3}{2.\ 3.\ 4b^2} - \frac{8x^4}{2.\ 3.\ 4.\ 5b^3} \&c.}\Big|^{-\frac{1}{2}} \times$$

$\overline{2A}|^{\frac{1}{2}} \times x^{\frac{1}{2}}\ \dot{x}$ *: which being compared with $\overline{a - bx^n + dx^{2n} + ex^{3n}\ \&c.}|^m \times x^{pn-1}\dot{x}$ (*vide Corol.* 2.) we ſhall, in this Caſe, have $n = 1$, $m = -\frac{1}{2}$, $p = \frac{1}{2}$, $a = a$, $b = \overline{1 \times \frac{2a}{b}} \times \frac{1}{2}$, $\frac{d}{b} = \frac{2}{3b}$, $\frac{e}{b} = -\frac{1}{3b^2}$, $\frac{f}{b} = \frac{2}{15b^3}$ &c. Whence $R = \frac{1}{3b}$, $S = \frac{2}{9b^2}$, $T = \frac{8}{45b^3}$ &c. Alſo $\frac{a^{p+m}P}{nb^p} \times \frac{1.3.5.7\ (p - \frac{1}{2}) \times 1.3.5.7\ (m + \frac{1}{2})}{2.4.6.8.10.\ (p + m)} = \frac{P}{b^{\frac{1}{2}}}$, and

$$1 + \frac{p+1.Ra}{p+m+1.b} + \frac{p+1.p+2.\ Sa^2}{p+m+1.p+m+2.\ b^2}\ \&c. = 1 + \frac{a}{2bb} + \frac{5a^2}{12b^2b^2}$$

* The Inveſtigation of this, and the Fluxion in the following Example, are both given in my Eſſays.

$+\frac{5a^2}{12b^2h^2}+\frac{7a^3}{18b^3h^3}$ &c. Whence we have $\overline{2A}|^{\frac{1}{2}} \times \frac{P}{b^{\frac{1}{2}}} \times \overline{1+\frac{a}{2bh}+\frac{5a^2}{nb^2h^2}}$, &c. for the Time of one Vibration of the Pendulum; which, by substituting $\overline{1+\frac{2a}{h}}\times\frac{1}{2}$ for its Equal b, &c. becomes $PA^{\frac{1}{2}} \times \overline{1 * +\frac{a^2}{6h^2}-\frac{2a^3}{9h^3}}$ &c. From which it appears, that the Effect of the Resistance on the Time of Vibration, in small Arches, is nearly in the duplicate Ratio of those Arches.

Sir *Isaac Newton* (from whom it is impossible to disagree without being under some Apprehensions of a Mistake) has, indeed, given a very different Solution to this Problem (in *Princip. Prop.* 27. *B.* 2.). But as the Conclusion here brought out exactly agrees with what I have elsewhere given, by a different Method, I have great Reason to believe I have no-where fallen into an Error.

The second Example I shall give as an Illustration of the foregoing Method is,

To determine the Apside Angle (or the Angle of the two Apses at the Center) in an Orbit described by means of a centripetal Force, which varies according to any Power of the Distance.

In order to which, let the Velocity of the Body at the higher Apse be to that whereby it might describe a Circle at the same Distance from the Center, in the given Ratio of p to Unity; also let

 that

that Diſtance be denoted by Unity; and, ſuppoſing z to denote any other Diſtance, let the centripetal Force be univerſally expreſſed by z^n. Then the Fluxion of the Angle at the Center will be expreſſed by $\frac{-p\dot{z}}{z\sqrt{\overline{p^2+\frac{2}{n+1}}\times z^2-p^2-\frac{2z^{n+3}}{n+1}}}$ Put $a=1-p^2$, $v=\frac{n+3}{2}$ and $x=1-z^2$, and it will become

$$\frac{\frac{1}{2}\sqrt{1-a}\times\dot{x}}{\overline{1-x}\times\sqrt{ax+\frac{1-vx-\overline{1-x}|^v}{1-v}}}=\tfrac{1}{2}\,\overline{1-a}|^{\frac{1}{2}}\text{ into}$$

$$\overline{a-\frac{vx}{2}+\frac{v.\,v-2}{2.\,3}\times x^2-\frac{v.\,v-2.\,v-3}{2.\,3.\,4}\times x^3\,\&c.}\Big|^{-\frac{1}{2}}\times\overline{x^{-\frac{1}{2}}\dot{x}+x^{\frac{1}{2}}\dot{x}+x^{\frac{3}{2}}\dot{x}\,\&c.}$$

Now, to find the Fluent of the firſt Term hereof (drawn into the general Multiplicator) or $\overline{a-\frac{vx}{2}+\frac{v.\,v-2}{2.3.}\times x^2\ \&c.}\Big|^{-\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}$, we have (as before) $n=1$, $m=-\frac{1}{2}$, $p=\frac{1}{2}$, $b=\frac{v}{2}$, $\frac{d}{b}=\frac{v-2}{3}$, $\frac{e}{b}=-\frac{v-2.\,v-3}{3.4}$, &c. Alſo $R=\frac{v-2}{6}$, $S=\frac{v-2.\,4v-5}{72}$; and conſequently the Fluent itſelf (when the Body arrives at the lower Apſe) $=\frac{P}{\sqrt{\frac{1}{2}v}}\times$

$$\overline{1+\frac{v-2}{2v}\times a+\frac{5.\,v-2.\,4v-5}{48v^2}\times a^2+\frac{7.\,v-v.\,16v^2-37v+22}{6.\,48v^3}}$$

&c. After the ſame manner the Fluent of the ſecond the second Term will come out $= \frac{P}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{a}{v} + \frac{5\cdot\overline{v-2}}{4v^2} \times a^2 + \frac{3\cdot\overline{v-2}\cdot\overline{2v-3}}{48v^3} \times a^3}$ &c. that of the third $= \frac{P}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{3a^2}{2v^2} + \frac{35\cdot\overline{v-2}}{12v^3} \times a^3}$ &c. &c. &c. Whence, by collecting these several Fluents together, we have $\frac{P}{\sqrt{\frac{1}{2}v}} \times$

$$\overline{1 + \tfrac{1}{2}a + \frac{20v^2-5v+2}{48v^2} \times a^2 + \frac{112v^3-53v^2+42v-8}{6.48v^3} \times a^3 \text{ \&c.}}$$

for the Fluent of the whole Expression: And this, drawn into $\frac{1}{2} \times \overline{-1 - \frac{a}{2} - \frac{a^2}{8}}$ &c. $(= \frac{1}{2} \times \overline{1-a}|^{\frac{1}{2}})$ will be

$$= \frac{P}{\sqrt{2v}} \times \overline{1 \; * \; \frac{\overline{v-2}.\overline{2v-1}}{48} \times \frac{a^2}{v^2} + \frac{\overline{v-2}.\overline{2v-1}|^2}{72} \times \frac{a^3}{v^3} \text{ \&c.}}$$

$$= \frac{P}{\sqrt{n+3}} \times \overline{1 \; * \frac{\overline{n-1}.\overline{n+2}}{24} \times \frac{a^2}{\overline{n+3}|^2} + \frac{\overline{n-1}.\overline{n+2}|^2}{18} \times \frac{a^3}{\overline{n+3}|^3} \text{ \&c.}}$$

which, in Degrees, gives $\frac{180}{\sqrt{n+3}} \times$

$\overline{1 \; * + \frac{\overline{n-1}.\overline{n+2}}{24} \times \frac{a^2}{\overline{n+3}|^2} + \frac{\overline{n-1}.\overline{n+2}|^2}{18} \times \frac{a^3}{\overline{n+3}|^3} \text{ \&c.}}$ for the true Measure of the Angle required.

 XIII.